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LETTER TO THE EDITOR

# Critical behaviour of the Baxter model with impurity lattice bonds

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**Abstract.** From one point of view, the Baxter model is a model of two coupled 2D Ising lattices. It is known that the specific heat exponent $\alpha$ in this model is proportional to the coupling between the lattices $g$, for small coupling: $\alpha \approx 4g/\pi \ll 1$. In this letter the critical behaviour of the specific heat of the Baxter model with impurity lattice bonds is studied and it is found that the Harris criterion does not hold in this case. In particular for $\alpha \sim g < 0$ the critical specific heat of the pure model $C_{\text{pure}}(\tau) \sim -|\tau|^{4(g)/\pi}[\tau = (T-T_c)/T_c]$ changes to a function with stronger cusp singularity $C_{\text{imp}}(\tau) \sim -(\ln \ln 1/|\tau|)^{-1}$, while according to the Harris criterion it should not change in this case. For $\alpha \sim g > 0$ the change is from $C_{\text{pure}}(\tau) \sim |\tau|^{-4g/\pi}$ to $C_{\text{imp}}(\tau) \sim \ln \ln 1/|\tau|$.

The problem of phase transitions in weakly disordered systems has received considerable attention in recent years. In the course of theoretical and experimental investigation, substantial progress has been achieved. In the papers by Harris and Lubensky (1974), Khmelnitsky (1975), Lubensky (1975), Grinstein and Luther (1976) it was shown that the singularities in thermodynamic functions near the phase transition point are not necessarily 'rounded', but a new critical behaviour can be established which differs from that of the pure system.

By general arguments, Harris (1974) had shown that the critical behaviour of systems with specific heat critical exponent $\alpha_{\text{pure}} < 0$ (specific heat is finite at $T_c$) is not affected by impurities. The critical behaviour is expected to be modified by the randomness only in systems with divergent specific heat ($\alpha_{\text{pure}} > 0$). This criterion has been backed by $\varepsilon$-expansion studies near 4D. It was found that in the case of $\alpha > 0$ the new critical exponents are universal, dependent only on the number of spin components and dimensionality. Moreover, the new specific heat exponent $\alpha_{\text{imp}}$ appeared to be negative, i.e. the specific heat was found to be finite at $T_c$ due to impurities.

Exact logarithmic singularities for the Ising model with impurities in 4D (and dipolar Ising model in 3D) have been found by Aharony (1976). The specific heat was found to be finite, having a non-analytic cusp behaviour at $T_c$. The 3D Ising model with impurities has been studied in detail by Newman and Riedel (1982) by an approximation technique. Their result $\alpha_{\text{imp}} \approx -0.09$ (cf $\alpha_{\text{pure}} \approx +0.11$) has been recently confirmed (within errors) by an experiment on real magnetic crystals (Birgeneau *et al* 1983).

Therefore a widespread belief now exists that the Harris' criterion and the finiteness of the specific heat are rather general properties of systems with randomness.

The 2D Ising model with impurities was of special interest since the pure system has an exact solution (Onsager 1944) and its specific heat critical exponent is zero.

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The critical behaviour of this model has been found by the authors (Dotsenko and Dotsenko 1982). It was shown that the specific heat remains divergent, though in a weaker way: $C_{\text{imp}}(\tau) - \ln\ln 1/|\tau|$ $[\tau = (T - T_c)/T_c]$, as compared to Onsager's singularity $C_{\text{pure}}(\tau) \sim \ln 1/|\tau|$.

The Baxter model presents an interesting case in these studies. This model, which has been solved exactly by Baxter (1971), can be considered as two 2D Ising models coupled by four-spin interactions (Kadanoff and Wegner 1971, Wu 1971, see also Baxter 1978). The strength of this coupling is described by some parameter $g$ (the case of $g = 0$ corresponds to two independent 2D Ising models). The critical exponents of the Baxter model are continuously dependent on $g$, and the specific heat exponent, which is proportional to $g$ for small coupling ($g \ll 1$), can be made both positive and negative. The aim of the present letter is to find the critical behaviour of the Baxter model with small randomness, keeping $g$ as a parameter, and compare the results with the general statements mentioned above.

Because the scaling limit of the Baxter model near the critical point is described by the 2D fermion (Thirring) model with four-fermion interaction (Luther and Peschel 1975, Luther 1976) the renormalisation group methods similar to those used for the 2D Ising model with impurities (Dotsenko and Dotsenko 1983) can also be applied to this case. The results are rather unexpected. For $g > 0$, when the specific heat of the pure model is divergent ($C_{\text{pure}}(\tau) \sim |\tau|^{-4g/\pi}$), the specific heat of the model with impurities is still divergent, but logarithmically, $C_{\text{imp}}(\tau) \sim \ln\ln 1/|\tau|$. On the other hand, for $g < 0$ ($C_{\text{pure}}(\tau) \sim -|\tau|^{4|g|/\pi}$ is finite) the specific heat of the model with impurities does not remain the same, as one would expect on the basis of the Harris criterion, but changes to a stronger cusp singularity: $C_{\text{imp}}(\tau) \sim -(\ln\ln 1/|\tau|)^{-1}$. Therefore the Harris criterion is not valid for the Baxter model with impurity bonds.

In the representation of two coupled 2D Ising lattices, the Baxter model is described by the following classical energy (see e.g. Baxter 1978)

$$H_0 = -\sum_{(x,x')} T_{x,x'}\sigma_x\sigma_{x'} - \sum_{(y,y')} T_{y,y'}\mu_x\mu_{x'} - T_4 \sum_{(x,x',y,y')} \sigma_x\sigma_{x'}\mu_x\mu_{x'}. \tag{1}$$

Here $\sigma$, $\mu = \pm 1$ are Ising variables. The first two terms correspond to two Ising models, and the sums go over nearest neighbours on two different interpenetrating square lattices. The third term is responsible for the coupling of the two models.

In the scaling limit near the critical point the homogeneous Baxter model ($T_{x,x'} = T_{y,y'} = T$) can equivalently be described by one complex (Dirac) or two real (Majorana) fermion fields (each one having two spinor components) with the Euclidian action

$$A = \int d^2x \left[ -\tfrac{1}{2}(\bar{\psi}\hat{\partial}\psi + \bar{\chi}\hat{\partial}\chi) - \frac{m_1}{2}\bar{\psi}\psi - \frac{m_2}{2}\bar{\chi}\chi + g_0(\bar{\psi}\psi)(\bar{\chi}\chi)\right]. \tag{2}$$

Here $\hat{\partial} = \gamma_\mu\partial_\mu = \gamma_1\partial_1 + \gamma_2\partial_2$, $\gamma_1 = \sigma^x$, $\gamma_2 = \sigma^z$, $\gamma^5 = \gamma^1\gamma^2 = -i\sigma^y$; $\sigma^x$, $\sigma^y$, $\sigma^z$ are Pauli matrices; $\bar{\psi} = \psi^T\gamma^5$, $\bar{\chi} = \chi^T\gamma^5$; $m_0 \sim \tau = (T = T_c)/T_c$; $g_0 \approx 2T_4$ for $T_4 \ll 1$ (see Luther and Peschel 1975). Quenched fluctuations of $T_{x,x'}$ and $T_{y,y'}$ can be described, in the scaling limit, by quenched Gaussian fluctuations of $\psi$ and $\chi$ masses in the action (2) (see Dotsenko and Dotsenko 1983)

$$A = \int d^2x \left( -\frac{1}{2}\bar{\psi}\hat{\partial}\psi = \frac{1}{2}\bar{\chi}\hat{\partial}\chi - \frac{m_0 + \delta m_1(x)}{2}\bar{\psi}\psi - \frac{m_0 + \delta m_2(x)}{2}\bar{\chi}\chi + g_0(\bar{\psi}\psi)(\bar{\chi}\chi)\right). \tag{3}$$

Here

$$\overline{\delta m_a(x)\delta m_b(x')} = 4\Delta_0\delta_{a,b}\delta(x-x'). \tag{4}$$

The constant $\Delta_0 \sim (\overline{T^2}-(\bar{T})^2)/(\bar{T})^2$ describes the quenched bond fluctuations, and it is assumed to be small. For the model with impurity bonds, randomly distributed over the lattice, it implies that the concentration of impurities is small.

After averaging the free energy over the randomness, the effective theory is described by the following replicated action (cf Dotsenko and Dotsenko 1983)

$$H = \int d^2x \left[ \sum_{A=1}^{N} \left( -\tfrac{1}{2}(\bar{\psi}^A\hat{\partial}\psi^A + \bar{\chi}^A\hat{\partial}\chi^A) - \frac{m_0}{2}(\bar{\psi}^A\psi^A + \bar{\chi}^A\chi^A) + g_0(\bar{\psi}^A\psi^A)(\bar{\chi}^A\chi^A) \right) + \frac{\Delta_0}{2} \sum_{A_1B=1}^{N} [(\bar{\psi}^A\psi^A)(\bar{\psi}^B\psi^B) + (\bar{\chi}^A\chi^A)(\bar{\chi}^B\chi^B)] \right]. \tag{5}$$

In the final results one should put $N=0$ (de Gennes 1972, Emery 1975). The theory (5) can be studied by renormalisation group methods. In the course of renormalisation an additional vertex appears

$$\gamma \sum_{A,B=1}^{N} (\bar{\psi}^A\psi^A)(\chi^B\chi^B) \tag{6}$$

($\gamma_0 = 0$). One can easily derive the following renormalisation group equations for the model (5), (6):

$$\begin{aligned} dg/d\xi &= -(4/\pi)g\Delta \\ \frac{d\Delta}{d\xi} &= -\frac{2(2-N)}{\pi}\Delta^2 + \frac{2N}{\pi}\gamma^2 + \frac{4}{\pi}g\gamma \\ \frac{d\gamma}{d\xi} &= \frac{4}{\pi}g\Delta - \frac{4(1-N)}{\pi}\Delta\gamma \\ \frac{d\ln m}{d\xi} &= -\frac{2(1-N)}{\pi}\Delta + \frac{2N}{\pi}\gamma + \frac{2}{\pi}g. \end{aligned} \tag{7}$$

The initial conditions are $g(0) = g_0$, $\Delta(0) = \Delta_0$, $\gamma(0) = 0$; $\xi$ is the renormalisation group parameter. For $N=0$ and $\xi = \frac{1}{4}\pi x$ the equations become

$$dg/dx = -g\Delta, \qquad d\Delta/dx = -\Delta^2 + g\gamma, \qquad d\gamma/dx = -\gamma\Delta + g\Delta \tag{8}$$

$$d\ln m/dx = -\Delta/2 + g/2. \tag{9}$$

Combining these equations one obtains

$$\gamma = g\ln(g_0/g), \qquad \Delta^2 = g^2(\ln^2(g_0/g) + \Delta_0^2/g_0^2). \tag{10}$$

The critical behaviour of thermodynamic functions is determined by the asymptotic behaviour of the solutions at $\xi \to \infty$. To find the relevant terms it is enough to replace the second relation in (10) by

$$\Delta \approx |g|\ln(g_0/g). \tag{11}$$

From (8) and (11) we obtain

$$dg/dx \approx -g|g|\ln(g_0/g), \qquad d\Delta/dx = -\Delta^2 + g|g|\ln(g_0/g). \tag{12}$$

The asymptotic solutions are given by

$$g \approx \frac{g_0}{t \ln t}\left(1 - \frac{\ln \ln t}{\ln t} + \frac{1}{\ln t}\right), \qquad \Delta \approx \frac{|g_0|}{t}\left(1 + \frac{1}{\ln t}\right). \tag{13}$$

Here $t = g_0 x = (4g_0/\pi)\xi$. Now from (9) and (13) we derive

$$\frac{\mathrm{d} \ln m}{\mathrm{d} t} = -\frac{1}{2t} - \frac{1}{2t \ln t}\left(1 - \frac{g_0}{|g_0|}\right)$$

$$\frac{m(t)}{m_0} \approx \begin{cases} \dfrac{1}{\sqrt{t}}, & g_0 > 0 \\ \dfrac{1}{\sqrt{t} \ln t} & g_0 < 0. \end{cases} \tag{14}$$

The singular part of the specific heat is given by (cf. Larkin and Khmelnitskii 1969)

$$C(\tau) \sim \int_0^{\ln(1/|\tau|)} \mathrm{d}\xi \cdot \left(\frac{m(\xi)}{m_0}\right)^2 \sim \frac{\pi}{4|g_0|} \int_0^{(4g_0/\pi)\ln(1/|\tau|)} \mathrm{d}t \left(\frac{m(t)}{m_0}\right)^2. \tag{15}$$

Using (14) we obtain

$$C_{\mathrm{imp}}(\tau) \sim \begin{cases} \dfrac{\pi}{4g_0} \ln \ln \dfrac{1}{|\tau|}, & g_0 > 0 \qquad (16) \\ -\dfrac{\pi}{4|g_0|}\left(\ln \ln \dfrac{1}{|\tau|}\right)^{-1} & g_0 < 0. \qquad (17) \end{cases}$$

As usual for statistical systems with small disorder, the new critical behaviour (16), (17) is established in the narrow temperature interval near $T_c$

$$|\tau| \ll \tau_i(g_0, \Delta_0) = \begin{cases} \left(\dfrac{\Delta_0}{|g_0|}\right) \pi/4|g_0|, & \Delta_0 \ll |g_0| \\ \exp\left[-\dfrac{\pi}{4|g_0|} \exp\left(\dfrac{\Delta_0}{|g_0|}\right)\right], & \Delta_0 \gtrsim |g_0|. \end{cases} \tag{18}$$

Note that the results are sensitive whether the random distribution of the spin couplings $\{T_{x,x'}, T_{y,y'}\}$ in (1) is independent of the two Ising lattices or whether it is not. The critical behaviour of the Baxter model with completely correlated distributions (i.e. $\overline{\delta m_1(x)\delta m_2(x')} = \overline{\delta m_1(x)\delta m_1(x')} = \overline{\delta m_2(x)\delta m_2(x')}$) turns out to be different. In this case the model is described by the replicated action (5) with the random vertex

$$\frac{\Delta_0}{2} \sum_{A,B=1} (\bar{\psi}^A \psi^A + \bar{\chi}^A \chi^A)(\bar{\psi}^B \psi^B + \bar{\chi}^B \chi^B). \tag{19}$$

It means that in equations (7) one has to take $\Delta(0) = \gamma(0) = \Delta_0$. The critical specific heat for this kind of Baxter model with impurities is found to be given by

$$C_{\mathrm{imp}}(\tau) \sim \begin{cases} \dfrac{\pi}{4g_0} \ln \ln \dfrac{1}{|\tau|}, & g_0 > 0 \qquad (20) \\ -\dfrac{\pi}{4|g_0|} |\tau|^{4|g^*|/\pi}, & g^* = g_0 \exp(-\Delta_0/|g_0|),\ g_0 < 0. \qquad (21) \end{cases}$$

In the case of $g_0 < 0$ the specific heat remains finite, but the critical exponent is altered due to impurities in a non-universal way i.e. $\alpha_{\rm imp} \sim g^*$ depends on the amount of randomness $\Delta_0$ (concentration of impurity bonds).

Note, however, that this kind of critical behaviour corresponds to a very special case of complete correlation in the bond distributions on the two lattices: $\overline{\delta m_1 \delta m_1} = \overline{\delta m_1 \delta m_2}$. It is absolutely unstable with respect to the introduction of any difference $\overline{\delta m_1 \delta m_1} - \overline{\delta m_1 \delta m_2} > 0$. One can check that the Baxter model with a partial correlation of the bond distributions ($\overline{\delta m_1 \delta m_1} > \overline{\delta m_1 \delta m_2} > 0$) exhibits, as $\tau \to 0$ (after some cross-over region), the same critical behaviour (16), (17) as the model with $\overline{\delta m_1 \delta m_2} = 0$ considered above.

In this letter we have found the critical specific heat of the Baxter model with impurity bonds—formulae (12) and (13). In a certain sense it is 'more universal' than that of the homogeneous model since it depends only on the sign of the coupling $g_0$. On the positive side ($\alpha_{\rm pure} \sim g_0 > 0$) the specific heat remains divergent, though in a much weaker way, as $\ln \ln 1/|\tau|$, the same as in the Ising Model with impurities (Dotsenko and Dotsenko 1982). On the negative side ($\alpha_{\rm pure} \sim g_p < 0$) the specific heat remains finite but the cusp singularity $[-(\ln \ln 1/|\tau|)^{-1}]$ is stronger than that in the pure model.

The quantitative result obtained here is that the Harris criterion does not hold in the present case. Whether this breakdown of the Harris criterion is a peculiar property of the Baxter model, or whether it is a rather general feature of the 2D statistics, remains an open question.

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